SELF-ADJOINT MODALITIES IN MTT

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Abstract. In this note, we record several results about the behavior of adjoint modalities in MTT. In particular, we show that internal adjunctions can be used to recover stronger rules, similar to Birkedal et al. [Bir+20]. As a use-case, we observe that self-adjoint modalities such as the modality used by Riley, Finster, and Licata [RFL21] admit similar rules to the handcrafted calculi presenting them, albeit with weaker definitional equalities.

1. Internal adjoints induce weak CwF morphisms

Let us consider the mode theory with two modalities $\mu : n \longrightarrow m$ and $\nu : m \longrightarrow n$ together with $\nu \dashv \mu$. In particular, there are 2-cells $\eta : \mathsf{id}_m \longrightarrow \mu \circ \nu$ and $\epsilon : \nu \circ \mu \longrightarrow \mathsf{id}_n$ such that

(1)
$$\begin{tikzcd} \mu \arrow[r, "\eta \star \mathsf{id}_\mu"] \arrow[dr, "\mathsf{id}_\mu"'] & \mu \circ \nu \circ \mu \arrow[d, "\mathsf{id}_\mu \star \epsilon"] \\ & \mu \end{tikzcd}$$

(2)
$$\begin{tikzcd} \nu \arrow[r, "\mathsf{id}_\nu \star \eta"] \arrow[dr, "\mathsf{id}_\nu"'] & \nu \circ \mu \circ \nu \arrow[d, "\epsilon \star \mathsf{id}_\nu"] \\ & \nu \end{tikzcd}$$

This mode theory is considered to some extent in Gratzer et al. [Gra+20], and it is shown there that ν behaves like a left adjoint internal to MTT. Here, we show that the right adjoint enjoys much of the behavior of a *dependent right adjoint* proper [Bir+20].

We now wish to prove the following result:

Theorem 1. *For any context* $\Gamma\ \mathsf{cx} \mathbin{@} m$ *and* $\Gamma \vdash A \mathbin{@} m$ *in a mode theory with* $\nu \dashv \mu$*:*

$$\Gamma.(\mathsf{id}_n \mid A).\{\mu\} \cong \Gamma.\{\mu\}.(\nu \mid A^\eta)$$

Proof. First, we observe that because $-.\{-\}$ is a 2-functor, it preserves (2-categorical) adjoints. Therefore, the $-.\{\nu\} \dashv -.\{\mu\}$. Note, however, that the change in variance causes the unit of this adjunction to be realized by $🔑^{\epsilon}$ and the counit to be realized by $🔑^{\eta}$.

Date: Saturday 17[th] April, 2021.

We start by constructing a substitution $\Gamma.(\mathsf{id}_n \mid A).\{\mu\} \longrightarrow \Gamma.\{\mu\}.(\nu \mid A^\eta)$:

$$(\uparrow.\{\mu\}).\mathbf{v}_0^\eta : \Gamma.(\mathsf{id}_n \mid A).\{\mu\} \longrightarrow \Gamma.\{\mu\}.(\nu \mid A^\eta) \mathrel{@} n \tag{3}$$

To explicitly write out the typing obligation, we must have $\Gamma.(\mathsf{id}_n \mid A).\{\mu \circ \nu\} \vdash \mathbf{v}_0^\eta : A[\uparrow \circ \key^\eta] \mathrel{@} m$.

The inverse morphism is slightly more difficult:

$$\Gamma.\{\mu\}.(\nu \mid A^\eta) \longrightarrow \Gamma.(\mathsf{id}_n \mid A).\{\mu\}$$

The complication is that $-.\{\mu\}$ generally does not have a 'mapping-in' property. Here, however, we are saved by the fact that μ is a right adjoint. We therefore switch to defining a substitution $\Gamma.\{\mu\}.(\nu \mid A^\eta).\{\nu\} \longrightarrow \Gamma.(\mathsf{id}_n \mid A)$:

$$(\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0 : \Gamma.\{\mu\}.(\nu \mid A^\eta).\{\nu\} \longrightarrow \Gamma.(\mathsf{id}_n \mid A) \mathrel{@} m$$

Unfolding the action of transposition, our substitution is explicitly given by the following:

$$((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu\} \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)} \tag{4}$$

We must now show that these are mutually inverse:

$$\begin{aligned}
&(((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu\}) \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)} \circ ((\uparrow.\{\mu\}).\mathbf{v}_0^\eta) \\
&= (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu\}) \circ ((\uparrow.\{\mu\}).\mathbf{v}_0^\eta.\{\nu \circ \mu\}) \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0) \circ ((\uparrow.\{\mu\}).\mathbf{v}_0^\eta.\{\nu\})).\{\mu\} \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= ((\key^\eta \circ (\uparrow.\{\nu\}) \circ ((\uparrow.\{\mu\}).\mathbf{v}_0^\eta.\{\nu\})).\mathbf{v}_0[(\uparrow.\{\mu\}).\mathbf{v}_0^\eta.\{\nu\}]).\{\mu\} \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= ((\key^\eta \circ (\uparrow.\{\nu\}) \circ ((\uparrow.\{\mu\}).\mathbf{v}_0^\eta.\{\nu\})).\mathbf{v}_0^\eta).\{\mu\} \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= ((\key^\eta \circ (\uparrow.\{\mu \circ \nu\})).\mathbf{v}_0^\eta).\{\mu\} \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= ((\uparrow \circ \key^\eta).\mathbf{v}_0^\eta).\{\mu\} \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= (\key^\eta_{\Gamma.(\mathsf{id}_n \mid A)}).\{\mu\} \circ \key^\epsilon_{\Gamma.(\mathsf{id}_n \mid A).\{\mu\}} \\
&= \key^{\eta \star \mathsf{id}_\mu}_{\Gamma.(\mathsf{id}_n \mid A)} \circ \key^{\mathsf{id}_\mu \circ \epsilon}_{\Gamma.(\mathsf{id}_n \mid A)} \\
&= \mathsf{id}
\end{aligned}$$

Now for the other direction

$$\begin{aligned}
&((\uparrow.\{\mu\}).\mathbf{v}_0^\eta) \circ (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu\}) \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)} \\
&= [(\uparrow.\{\mu\}) \circ (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu\}) \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)}].\mathbf{v}_0[\key^\eta \circ (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu \circ \nu\}) \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)}.\{\nu\}] \\
&= [\key^\eta.\{\mu\} \circ \key^\epsilon_{\Gamma.\{\mu\}} \circ \uparrow].\mathbf{v}_0[\key^\eta \circ (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu \circ \nu\}) \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)}.\{\nu\}] \\
&= \uparrow.\mathbf{v}_0[\key^\eta \circ (((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu \circ \nu\}) \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)}.\{\nu\}] \\
&= \uparrow.\mathbf{v}_0[((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0) \circ \key^\eta \circ \key^\epsilon_{\Gamma.\{\mu\}.(\nu \mid A^\eta)}.\{\nu\}] \\
&= \uparrow.\mathbf{v}_0[((\key^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0)] \\
&= \uparrow.\mathbf{v}_0 \\
&= \mathsf{id}
\end{aligned}$$

∎

Theorem 2. *For any context $\Gamma \ \mathsf{cx} \mathrel{@} m$ and $\Gamma.\{\xi\} \vdash A \mathrel{@} o$ in a mode theory with $\nu \dashv \mu$:*

$$\Gamma.(\xi \mid A).\{\mu\} \cong \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \mathsf{id}_\xi})$$

Proof. The substitutions are similar to the ones from the 'unframed' variant:

$$(\uparrow.\{\mu\}).\mathbf{v}_0^{\eta \star \mathsf{id}_\xi} : \Gamma.(\xi \mid A).\{\mu\} \longrightarrow \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \mathsf{id}_\xi}) \mathbin{@} n$$

$$((\text{🔑}^\eta \circ \uparrow.\{\nu\}).\mathbf{v}_0).\{\mu\} \circ \text{🔑}^\epsilon_{\Gamma.\{\mu\}.(\nu\circ\xi \mid A^{\eta \star \mathsf{id}_\xi})} : \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \mathsf{id}_\xi}) \longrightarrow \Gamma.(\xi \mid A).\{\mu\} \mathbin{@} m$$

The calculations that these are inverse are functionally identical to those in Theorem 1. □

As a result of Theorem 2, we obtain the following derived introduction rule for μ:

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$$\frac{\mathbf{1}.(\nu \circ \xi_0 \mid A_0') \ldots .(\nu \circ \xi_n \mid A_n') \vdash M' : A' \mathbin{@} m}{\mathbf{1}.(\xi_0 \mid A_0). \cdots .(\xi_n \mid A_n) \vdash \mathsf{mod}_\mu(M) : \langle \mu \mid A \rangle \mathbin{@} m}$$

Here A_i' is obtained from A_i by applying the appropriate substitutions induced by Theorem 2 to ensure that the dependence of A_i on prior entries in the context is correctly aligned. Likewise, A' and M' are obtained from A and M by applying the same substitutions. Note that this rule is entirely derivable within MTT, not axioms or admissibilities are required.

We also note that under these assumptions μ is very nearly a full dependent right adjoint:

Theorem 3. *Given any context $\Gamma\ \mathsf{cx} \mathbin{@} m$ and $\Gamma.\{\mu\} \vdash A \mathbin{@} n$ when $\nu \dashv \mu$, there is a pair of substitutions*

$$\gamma^\to : \Gamma.(\mu \mid A) \longrightarrow \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \mathbin{@} m$$
$$\gamma^\leftarrow : \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \longrightarrow \Gamma.(\mu \mid A) \mathbin{@} m$$

Moreover, $\gamma^\leftarrow \circ \gamma^\to = \mathsf{id}$ and, if one assumes extensional equality, $\gamma^\to \circ \gamma^\leftarrow = \mathsf{id}$.

Proof. One direction of this isomorphism holds regardless of the precise properties of μ:

(5) $$\gamma^\to \triangleq \uparrow.\mathsf{mod}_\mu(\mathbf{v}_0) : \Gamma.(\mu \mid A) \longrightarrow \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \mathbin{@} m$$

The inverse direction is more subtle:

(6) $$\gamma^\leftarrow \triangleq \uparrow.M : \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \longrightarrow \Gamma.(\mu \mid A) \mathbin{@} m$$

Here, M must be a term of the following type:

$$\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\} \vdash M : A[\uparrow.\{\mu\}] \mathbin{@} n$$

In order to define this, consider the following term:

$$\frac{\begin{array}{c}\Gamma.(\mathsf{id}_n \mid \langle \mu \mid A \rangle).\{\mu \circ \nu\} \vdash \mathbf{v}_0^\eta : \langle \mu \mid A[\text{🔑}^{\eta \star \mathsf{id}_\mu}] \rangle \mathbin{@} m \\ \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\}.(\nu \circ \mu \mid A) \vdash \mathbf{v}_0^\epsilon : A[\uparrow.\{\nu\}] \mathbin{@} m\end{array}}{\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\} \vdash M \triangleq \mathsf{let}_\nu\ \mathsf{mod}_\mu(\_) \leftarrow \mathbf{v}_0^\eta\ \mathsf{in}\ \mathbf{v}_0^\epsilon : A[\uparrow.\{\mu\}] \mathbin{@} n}$$

By computation, we immediately have $\gamma^\leftarrow \circ \gamma^\to = \mathsf{id}$. In the reverse direction, we must show that the following terms are definitionally equivalent

(7) $$\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \vdash \mathbf{v}_0 = \mathsf{mod}_\mu(\mathsf{let}_\nu\ \mathsf{mod}_\mu(\_) \leftarrow \mathbf{v}_0^\eta\ \mathsf{in}\ \mathbf{v}_0^\epsilon) : \langle \mu \mid A[\uparrow.\{\mu\}] \rangle \mathbin{@} m$$

This equation is true *propositionally*, by performing induction on $\mathbf{v}_0$. Therefore, in the presence of extensional equality this holds definitionally as well. □

From this, we conclude that (up to issues of commuting conversions) an internal right adjoint in MTT is precisely a dependent right adjoint.

2. Self-adjoint modalities

We now return to the motivation for this note. In Riley, Finster, and Licata [RFL21] a theory of synthetic spectra is developed around a modality $\natural$ which is not only a monad and comonad, but it is self-adjoint $\natural \dashv \natural$. From this, we may replay the results of Section 1 to conclude that there is an isomorphism

$$\Gamma.(\natural^n \mid A).\{\natural\} \cong \Gamma.\{\natural\}.(\natural^{n+1} \mid A) \tag{8}$$

Iterating this, it appears one may derive both the introduction and elimination rules for $\natural$ given in Riley, Finster, and Licata [RFL21] from the standard MTT rules. The definitional equalities proposed in this work, however, may be weakened to mere propositional equalities owing to results like Theorem 3. With the introduction of either appropriate commuting conversions or extensional equality, however, this too can be encoded

References

[Bir+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. "Modal dependent type theory and dependent right adjoints". In: *Mathematical Structures in Computer Science* 30.2 (2020), pp. 118–138. DOI: 10.1017/S0960129519000197 (cit. on p. 1).

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