# CRISP INDUCTION FOR INTENSIONAL IDENTITY TYPES

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ABSTRACT. We investigate the interaction between modal types and identity types in MTT and emphasize the connection to function extensionality. We show that the desired "extensionality" principle holds in extensional MTT and conjecture the same for cubical MTT. Finally, we show that extending MTT with 'crisp' induction principles yields the same extensionality principles in MTT without disrupting canonicity or normalization.

CONTENTS



Throughout this note we work in MTT with a fixed mode theory $\mathcal{M}$ containing at least a modality $\mu : n \longrightarrow m$.

*Notation* 1. When working informally in MTT, we write $\{\mu\} \vdash \ldots$ to denote a judgment taking place in the ambient context locked by $\mu$.

## 1. MODAL IDENTITY TYPES

The central question is the relationship between the identity type of modal types $\mathsf{Id}_{\langle \mu \mid A \rangle}(\mathsf{mod}_\mu(M), \mathsf{mod}_\mu(N))$ and the modalized identity type $\langle \mu \mid \mathsf{Id}_A(M, N) \rangle$. In particular, we are interested in the following equivalence

$$\mathsf{Id}_{\langle \mu \mid A \rangle}(\mathsf{mod}_\mu(M), \mathsf{mod}_\mu(N)) \simeq \langle \mu \mid \mathsf{Id}_A(M, N) \rangle \tag{1}$$

Unfortunately, we can quickly show that Eq. (1) cannot be defined in MTT.

**Theorem 1.1.** *The equivalence* $\mathsf{Id}_{\langle \mu \mid A \rangle}(\mathsf{mod}_\mu(M), \mathsf{mod}_\mu(N)) \simeq \langle \mu \mid \mathsf{Id}_A(M, N) \rangle$ *is independent of MTT*

*Proof.* It is trivial to construct models of MTT where this equivalence is satisfied (interpret all modalities with the identity for instance). It remains to construct a model where it is false. Consider the model of MTT where all modes are interpreted by the same category: the syntactic model of MLTT. Let us then interpret each modality as exponentiation by a closed types, chosen so that 2-cells between modalities can be realized by maps between these closed types.

In this model, the equivalence under consideration unfolds to the following:

$$\mathsf{Id}_{\mathfrak{B} \to A}(\lambda(M), \lambda(N)) \simeq (\mathfrak{b} : \mathfrak{B}) \to \mathsf{Id}_A(M(\mathfrak{b}), N(\mathfrak{b}))$$

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This is the axiom of function extensionality, which is well-known to be independent of $\mathsf{MLTT}$. Therefore, this equivalence cannot be interpreted into this model, thus it cannot be proven in $\mathsf{MTT}$. □

We obtain three important ideas from the connection between Eq. (1) and function extensionality exploited in Theorem 1.1. The first is the name 'modal extensionality' for Eq. (1). The second, perhaps more useful, is the intuition we should have at least one direction of Eq. (1). The final insight is that the solutions to function extensionality should have some bearing on Eq. (1).

**Theorem 1.2.** *There is a canonical map* $\rho : \mathsf{Id}_{\langle \mu | A \rangle}(\mathsf{mod}_\mu(M), \mathsf{mod}_\mu(N)) \to \langle \mu \mid \mathsf{Id}_A(M, N) \rangle$

*Proof.* While intuitively the proof is simply "induct and use reflexivity" we take the time to write out this term with some care. In particular, we specify the motive of this induction properly:

$$C : (x_0, x_1 : \langle \mu \mid A \rangle) \to \mathsf{Id}_{\langle \mu | A \rangle}(x_0, x_1) \to \mathsf{U}$$
$$C = \lambda x_0, x_1, \_. \ \mathsf{let}\ \mathsf{mod}_\mu(x_0) \leftarrow m_0 \ \mathsf{in}\ \mathsf{let}\ \mathsf{mod}_\mu(x_1) \leftarrow m_1 \ \mathsf{in}\ \langle \mu \mid \mathsf{Id}_A(x_0, x_1) \rangle$$

The actual inductive argument is then quite straightforward:

$$\rho = \mathsf{J}(C, x.\mathsf{let}\ \mathsf{mod}_\mu(x') \leftarrow x \ \mathsf{in}\ \mathsf{mod}_\mu(\mathsf{refl}(x')), -)$$

By computation, we see that $\rho(\mathsf{refl}(\mathsf{mod}_\mu(M))) = \mathsf{mod}_\mu(\mathsf{refl}(M))$. □

**Theorem 1.3.** *In the presence of equality reflection, $\rho$ is an isomorphism.*

*Proof.* We can easily define a candidate inverse $\bar{\rho}$ through equality reflection. To this end, we assume $(\mu \mid p : \mathsf{Id}_A(M, N))$, whereby we may prove $\{\mu\} \vdash M = N : A$ by reflection so the result follows by congruence. We immediately have $\bar{\rho} \circ \rho = \mathsf{id}$ by UIP, so it only remains to argue that $\rho \circ \bar{\rho} = \mathsf{id}$.

We prove this by showing that $\langle \mu \mid A \rangle$ is a proposition if $A$ is a proposition. Fix two elements of $\langle \mu \mid A \rangle$, which by induction are of the form $\mathsf{mod}_\mu(x)$ and $\mathsf{mod}_\mu(y)$ where $(\mu \mid x, y : A)$. By the assumption that $A$ is a proposition, we conclude that $\{\mu\} \vdash x = y : A$ whence $\mathsf{mod}_\mu(x) = \mathsf{mod}_\mu(y) : \langle \mu \mid A \rangle$. □

*Remark* 2. In fact, in many models of $\mathsf{MTT}$ the modalities are realized by full right adjoints. Therefore Theorem 1.3 is an incarnation of the fact that right adjoints preserve limits (equalizers in this case). Given the subtle differences between right adjoints and dependent right adjoints and compounding with the distinction between dependent right adjoints and $\mathsf{MTT}$ modalities, one should be not to take such arguments as a proof for $\mathsf{MTT}$ itself; it is only valid in well-behaved models.

## 2. Crisp induction principles

Eq. (1) has already been explored in prior modal type theories, including $\mathsf{MTT}$. For instance, Shulman [Shu18] and Gratzer, Kavvos, Nuyts, and Birkedal [Gra+21] show that Eq. (1) is validated by a modality with a right adjoint. These theorems are proven by a "crisp" induction principle which is an indirect way of stating that modalities preserve identity types.

Given that most models of $\mathsf{MTT}$ are extensional and therefore already satisfy Eq. (1) by Theorem 1.3, it is natural to wonder whether $\mathsf{MTT}$ could simply be directly extended by a crisp induction principle for identity types.

Concretely, we are interested in the following induction scheme:

$$\frac{\begin{array}{c}\mu : n \longrightarrow m \qquad \Gamma.\{\mu\} \vdash A \qquad \Gamma.(\mu \mid A).(\mu \mid A[\uparrow]).(\mu \mid \mathsf{Id}_{A[\uparrow^2]}(\mathbf{v}_1, \mathbf{v}_0)) \vdash B \\ \Gamma.(\mu \mid A) \vdash M : B[\uparrow.\mathbf{v}_0.\mathbf{v}_0.\mathsf{refl}(\mathbf{v}_0)] \\ \Gamma.\{\mu\} \vdash N_0, N_1 : A \qquad \Gamma.\{\mu\} \vdash P : \mathsf{Id}_A(N_0, N_1)\end{array}}{\Gamma \vdash \mathsf{J}^\mu(B, M, P) : B[\mathsf{id}.N_0.N_1.P]}$$

We also include the expected definitional equality: $\mathsf{J}^\mu(B, M, \mathsf{refl}(N)) = M(N)$.

First, we observe that this rule is sufficient to prove Eq. (1):

$$\bar{\rho}(\mathsf{mod}_\mu(p)) = \mathsf{J}^\mu(x_0, x_1, \_.\ \mathsf{Id}(\mathsf{mod}_\mu(x_0), \mathsf{mod}_\mu(x_1)), x.\mathsf{refl}(\mathsf{mod}_\mu(x)), p)$$

We can prove that $\bar{\rho}$ is an inverse through (crisp) induction. In fact, this rule is precisely equivalent to Eq. (1) (assuming the latter sends reflexivity proofs to reflexivity proofs).

**Theorem 2.1.** *A model of extensional* MTT *supports crisp identity induction.*

Given the connection between modal extensionality and function extensionality, we should regard this induction principle with healthy skepticism; it is sound but it seems likely that it would disrupt either canonicity or normalization. In fact, both of these properties continue to hold after the addition of this principle. This remarkable fact hinges on the relative paucity of elements of $\langle \mu \mid A \rangle$ compared to $\mathfrak{B} \to A$. In particular, there are *no* closed neutral elements of $\langle \mu \mid A \rangle$ in MTT without extending the system by some axioms. As a result, the computation rule $\mathsf{J}^\mu(B, M, \mathsf{refl}(N)) = M(N)$ is actually sufficient to ensure computational adequacy in the syntax.

2.1. **Normalization with primitive crisp induction.** We prove this fact by extending Gratzer [Gra21] to support crisp induction. The specific alterations to the proof are extremely minor. We first alter the neutral for identity elimination to this more general form as well as the definition of MTT cosmoi. We thereby are left with two new constants in the gluing topos:

$$\begin{aligned}
&\mathsf{J}_\mu : (\mu \mid A : \mathsf{Ty}_n)(B : (\mu \mid a_0, a_1 : \mathsf{Tm}_n(A))(\mu \mid p : \mathsf{Tm}_n(\mathsf{Id}(A, a_0, a_1))) \to \mathsf{Ty}_m) \to \\
&\qquad ((\mu \mid a : \mathsf{Tm}_n(A)) \to \mathsf{Tm}_m(B(a, a, \mathsf{refl}(a)))) \to \\
&\qquad (\mu \mid a_0, a_1 : \mathsf{Tm}_n(A))(\mu \mid p : \mathsf{Tm}_n(\mathsf{Id}(A, a_0, a_1))) \to \mathsf{Tm}_m(B(a_0, a_1, p)) \\
&\mathbf{J}_\mu : (\mu \mid A : \bigcirc\mathsf{Ty}_n)(B : (\mu \mid a_0, a_1 : \mathsf{V}_n(A))(\mu \mid p : \mathsf{V}_m(\mathsf{Id}(A, a_0, a_1))) \to \mathsf{NfTy}_m) \to \\
&\qquad ((\mu \mid a : \mathsf{V}_n(A)) \to \mathsf{Nf}_m(B(a, a, \mathsf{refl}(a)))) \to \\
&\qquad (\mu \mid a_0, a_1 : \bigcirc_z\mathsf{Tm}_n(z, A(z)))(\mu \mid p : \mathsf{Ne}_n(\mathsf{Id}(A, a_0, a_1))) \to \mathsf{Ne}_m(B(a_0, a_1, \eta(p)))
\end{aligned}$$

This change impacts the construction of the normalization proof in precisely one place: the normalization algebra for identity types. Specifically, we must change the construction of $\mathsf{J}^*_\mu$.

**Theorem 2.2.** *In the context of Section 7 of Gratzer [Gra21], we have a term of the following type:*

$$\begin{aligned}
&\mathsf{J}^*_\mu : (\mu \mid A : \mathsf{Ty}^*_n)(B : (\mu \mid a_0, a_1 : \mathsf{Tm}^*_n(A))(\mu \mid p : Tm * n(\mathsf{Id}^*(A, a_0, a_1))) \to \mathsf{Ty}^*_m) \to \\
&\qquad (b : (\mu \mid a : \mathsf{Tm}^*_n(A)) \to \mathsf{Tm}^*_m(B(a, a, \mathsf{refl}^*(a)))) \to \\
&\qquad (\mu \mid a_0, a_1 : \mathsf{Tm}^*_n(A))(\mu \mid p : \mathsf{Tm}^*_n(\mathsf{Id}(A, a_0, a_1))) \to \\
&\qquad \{\mathsf{Tm}^*_m(B(a_0, a_1, p)) \mid z : \mathbf{syn} \mapsto \mathsf{J}_\mu(A, B, b, p)\}
\end{aligned}$$

*Proof.* Let us fix $A$, $B$, $b$, $a_0$, $a_1$, and $p$ with the types described above. Recalling the definition of $\mathsf{pred}(\mathsf{Id}^*(A, a_0, a_1))$, we can commute $\langle \mu \mid - \rangle$ past the dependent sum, closed modalities, equality types[1], and coproducts to obtain an element of the following type:

$$\textstyle\sum_{m:\langle \mu \mid \mathsf{Nf}_n(\mathsf{Id}(A,a_0,a_1))\rangle} \bullet \Big[\Big(\sum_{e:\langle \mu \mid \mathsf{Ne}_n(\mathsf{Id}(A,a_0,a_1))\rangle} \mathbf{up}(e) = m\Big) + (a_0 = a_1 \times m = \mathsf{refl}(a_0))\Big]$$

Write $p' = \langle l, r \rangle$ for the image of $p$ under this isomorphism. We then define $\mathsf{J}^*_\mu(B, b, a_0, a_1, p)$ by analyzing $r$:

$$\begin{cases} \mathsf{J}(z, B, b, a_0, a_1, p) & q = \mathsf{in}_0(z) \\ \downarrow_{B(p)} \mathbf{J}(\lambda a_0, a_1, p.\ \mathsf{code}(T_m(\uparrow a_0, \uparrow a_1, \uparrow p)), \lambda a.\ \downarrow_{B(a,a,\mathsf{refl}^*(a))} b(\uparrow_A a), e) & q = \mathsf{in}_1(\mathsf{in}_0(e, \star)), \mathbf{up}(e) = p \\ b(a_0) & q = \mathsf{in}_1(\mathsf{in}_1(\star, \star)) \end{cases}$$ □

Finally, even after altering our neutral forms to include neutral crisp induction, we have maintained the property that there are no neutral elements in the context $\mathbf{1}.\{\mu\}$. Accordingly, the normalization result simultaneously establishes modal canonicity for MTT; the only normal forms in contexts of the form $\mathbf{1}.\{\mu\}$ are canonical forms.

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## References

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[1] Note that we are working in extensional MTT, so Theorem 1.3 applies.